Frequencies and energy levels of the weakly relativistic harmonic oscillator from the action variable

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Abstract

The frequency of a classical periodic system and the energy levels of the corresponding quantum system can both be obtained using action variables. We demonstrate the construction of two forms of the action variable for a one dimensional harmonic oscillator in classical, relativistic and quantum regimes. The relativistic effects are considered as perturbative, within the context of a non-relativistic quantum formalism. The transition of the relativistic quantum system to both classical relativistic and classical non-relativistic regimes is illustrated in a unified framework. Formulas for the frequency of a classical relativistic oscillator and the energy eigenvalues of the corresponding quantum oscillator for the weak relativistic case are derived. Also studied are the non-relativistic and classical limits of these formulas which provide valuable insights on the parallels between relativistic and non-relativistic systems on the one hand and between classical and quantum systems on the other.

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I. INTRODUCTION

Classical periodic systems can be analyzed elegantly in terms of their action and angle variables which constitute a set of canonically conjugate momenta and coordinates. Action variables are proportional to $\oint p_j dq_j$, where (q_j, p_j) are the system's coordinates and canonical momenta. For separable systems they are constants of motion in the manner of angular momentum and energy. The frequencies of periodic systems can be found using the functional relationship between the action variables and total mechanical energy without requiring a complete solution of the dynamical equations. The French astronomer and mathematician Charles-Eugène Delaunay (1816-1872) invented action and angle variables in the course of his study of periodicity of lunar motion[1]. These variables assumed importance during the early days of quantum mechanics. Lord Rayleigh had shown that in a sinusoidally oscillating system such as a pendulum whose string is shortened slowly, the ratio of the energy to frequency, which is directly proportional to the action variable, remains a constant. Following the language of thermodynamics such motion was referred to as "adiabatic" motion. At the first Solvay Conference in 1911, which considered the issues that the new quantum ideas introduced into mechanics, it was realized that the adaibatic invariance of the action variable in atomic systems, in an environment of slowly varying electromagnetic fields, would lead to atomic stability without transitions between states[2]. Schwarzschild introduced into quantum theory the analytical method of employing action variables[3]. The quantization rules of Sommerfeld, Wilson and Ishiwara required that the action variables be integer multiples of $\hbar$ to account for the energy spectra of atomic systems.[4,5,6]. Ehrenfest formulated the "adiabatic principle" according to which a slow variation of some parameters of a periodic system's Hamiltonian would result in a gradual change in the system's motion while maintaining the constancy of the action variable[7,8]. With the establishment of the wave and matrix forms of quantum mechanics the program of employing action variables in the quantum context did not receive significant attention. In the JWKB approximation scheme for the determination of bound quantum states the energy eigenvalues are obtained by discretizing the action variable. Thus the quantum conditions of the old quantum theory can be rigorously deduced as an approximate result in the new quantum theory.

In 1983 Leacock and Padgett[9,10] presented a form of quantum mechanics, patterned on the classical Hamilton-Jacobi theory and equivalent to the Schrdinger theory, whose focus

was a quantum version of the action variable which reduces to the traditional action variable in the classical limit. In this formalism the dynamical equation is a postulated quantum Hamilton-Jacobi equation for the Hamilton's principal function S which generates a transformation from the coordinates and momenta (x_i, p_i) to angle and action variables (w_i, J_i). The bound states of a system are characterized by its quantum action variables assuming values which are integral multiples of $\hbar$ in a natural way, and not through arbitrary imposition as was the case in the Wilson-Sommerfeld scheme. Since the total mechanical energy of the system is a function of its action variables it too assumes discrete values for bound states. The examination of the dynamical equations of this formalism of quantum theory, and the relation between quantum action variables and energy, shows the classical-quantum correspondence in a new light. A systematic development of canonical transformations and the Hamilton-Jacobi theory in quantum mechanics is found in Ref. 11. Many systems admitting bound states have been studied using this quantum Hamilton-Jacobi formalism[12,13]. Our objectives here are two fold. We show, using a perturbative approach based on two equivalent forms of the action variable, how a weakly relativistic quantum oscillator can be treated within the non-relativistic quantum Hamilton-Jacobi formalism to obtain its energy eigenvalues that incorporate first order relativistic corrections. Two, we show that the dynamics of this weakly relativitic oscillator has the correct non-relativistic classical and non-relativistic quantum limits.

The symbols and notation we use for physical variables have the following meaning. The angular frequency of a classical non-relativistic simple harmonic oscillator of mass m connected to a massless spring of spring constant k will be $\omega_0 = \sqrt{\frac{k}{m}}$. The suffix $_C$ refers to a classical variable; the absence of this suffix indicates that the quantity being referred to is its quantum counterpart. Similarly the suffix $_R$ refers to a relativistic variable, and $_{WR}$ is the suffix for a variable in the weak relativistic case.

II. ACTION VARIABLE IN CLASSICAL MECHANICS

A. Classical Hamilton-Jacobi Theory

The time evolution of a classical system is governed by its Hamiltonian H which is a function of its coordinates x_i, the conjugate momenta p_i and the time t. The dynamics of

such a system is determined by Hamilton's equations of motion

$$\dot{x}_i = \frac{\partial H(x_i, p_i, t)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(x_i, p_i, t)}{\partial x_i}. \tag{1}$$

The Hamiltonian of a particle of mass m moving in one dimension under the influence of a potential energy function $V(x)$ is given by $H = \frac{p^2}{2m} + V(x)$. Such a time independent Hamiltonian is a constant of the motion and is the total energy E of the system. Thus,

$$\frac{p^2}{2m} + V(x) = E. \tag{2}$$

Canonical transformations transform one set of coordinate and momentum (x, p) to another set (X, P) while preserving the form of Hamilton's equations. One such transformation is generated by the function $W_C(x, P)$, whose arguments are the "old" coordinate, x, and the "new" momentum, P:

$$p = \frac{\partial W_C(x, P)}{\partial x}, \;\; X = \frac{\partial W_C(x, P)}{\partial P}. \tag{3}$$

If this transformation transforms the Hamiltonian into a function only of P, then, using (1),

$$\begin{aligned} \dot{P} = -\frac{\partial H(P)}{\partial X} = 0 \;\Rightarrow \quad & P(t) = P, \text{ a constant}, \\ \dot{X} = \frac{\partial H(P)}{\partial P} = V_0, \;\; \text{a constant}, \qquad \Rightarrow \;\; & X(t) = V_0\, t + X_0. \end{aligned} \tag{4}$$

Thus X and P evolve very simply in time; the former has a linear temporal progress and the latter is a constant. The function $W_C(x, P)$, which generates a canonical transformation in which the transformed Hamiltonian is independent of the new coordinate X, is the Hamilton's *characteristic* function. It is related to the Hamilton's *principal* function S_C through $S_C(x, P, t) = W_C(x, P) - Et$, and, for the case of time independent Hamiltonians, satisfies the Hamilton-Jacobi equation obtained by using (3) in (2):

$$\frac{1}{2m}\left(\frac{\partial W_C(x, P)}{\partial x}\right)^2 + V(x) \;=\; E(P). \tag{5}$$

The use of this method to solve the dynamical problem involves the following steps: (i) Define a suitable new constant momentum P, (ii) Integrate Eq. (5) to obtain $W_C(x, E(P))$, (iii) Obtain $x(X, P)$ and $p(X, P)$ using Eq. (3), and (iv) Express X and P in terms of the initial values x_0, p_0 and t.

B. Action-angle variables and periodic motion

One particular form of Hamilton-Jacobi theory is especially suited to the study of periodic motion. If an inspection of the Hamiltonian indicates that the motion is periodic, then by a particular choice of the new momentum P we can evaluate the period of motion without obtaining a complete solution of the dynamical problem. The new canonically conjugate coordinate and momentum are chosen to be $X = w,\ P = J_C$ with

$$J_C \ = \ \frac{1}{2\pi} \oint p_C(x, E) dx, \tag{6}$$

where $p_C(x, E)$, from (2), is $\sqrt{2m[E - V(x)]}$ and the integral in phase space is performed over one cycle of the periodic motion. J_C is the classical action variable and w the angle variable. We note that the integral for J_C is the area enclosed in phase space by the path of the oscillator's orbit. A new momentum, similar to J_C, will be defined in the corresponding quantum formalism and will be referred to as J. Since $J_C = J_C(E)$ we can invert it to obtain $E = E(J_C)$. From Eq. (4) the time evolution of the new coordinate is $w(t) = \omega\, t + w_0$ where the constant "velocity" is

$$\omega = \frac{\partial H(J_c)}{\partial J_C} = \frac{\partial E(J_c)}{\partial J_C}. \tag{7}$$

It can be shown[14] that ω is the angular frequency of this periodic motion. Thus the mathematical problem of finding the frequency of motion for a periodic system is reduced to that of performing the integral (6), solving for E to get $E(J_C)$, and evaluating $\partial E/\partial J_C$. This is a simple and elegant method for evaluating the frequency of a system known to be periodic. An equally simple method can be used to obtain the energy eigenvalues of bound states in quantum mechanics.

An equivalent definition of J_C, which is useful for extending the action variable into the quantum arena, is

$$J_C \ = \ \frac{1}{2\pi} \oint_C p_C(x, E) dx, \tag{8}$$

where $p_c(x, E)$ is a complex valued function of the complex argument x, and is defined as a suitable branch of

$$p_c(x, E) \ = \ \sqrt{2m\,[E - V(x)]}. \tag{9}$$

The turning points x_1 and x_2 are defined by $p_c(x_1, E) = p_c(x_2, E) = 0$. These are also the branch points of $p_c(x, E)$ in the complex-x plane. We choose a branch cut connecting x_1 and x_2 along the real axis. $p_c(x, E)$ is chosen as that branch of the square root which is positive along the bottom of the cut. The counterclockwise rectangular contour C wraps around this branch cut. The integral in (8) is performed by deforming the contour C outward to the circular contour γ which lies in an annulus in which $p_c(x, E)$ is analytic, expanding $p_c(x, E)$ in a Laurent series in that annulus and integrating using Cauchy's residue theorem. Sommerfeld was the first to employ this contour integral technique in evaluating the action variable for the bound states of the electron in hydrogenic atoms.

An alternate construction of the action variable arises from another canonical transformation scheme where the alternate Hamilton's principal function $\tilde{S}_C(p, X, t)$ of the "old" momentum and "new" coordinate is used. For conservative systems, the canonical transformation generated by the alternate Hamilton's characteristic function $\tilde{W}_C(p, X)$, where $\tilde{S}_C(p, X, t) = \tilde{W}_C(p, X) - Et$, is $x = -\frac{\partial \tilde{W}_C(p,X)}{\partial p}$, $P = -\frac{\tilde{W}_C(p,X)}{\partial X}$. The Hamilton-Jacobi equation satisfied by this characteristic function is

$$H\left(-\frac{\partial \tilde{W}_C(p, X)}{\partial p}, p\right) = E. \tag{10}$$

As in the previous scheme, for periodic systems, we can choose the new coordinate as the alternate action variable $J_C = -\frac{1}{2\pi} \oint_{C'} x_C(p, E) dp$, where the clockwise rectangular contour C' in the complex-p plane surrounds the branch cut along the real axis connecting the turning momenta p_1 and p_2 and the sign of $x_C(p, E)$ is chosen positive below this cut. We will demonstrate the use of this alternate form of the action variable in both the classical and quantum contexts for the weakly relativistic harmonic oscillator. The angle variable, while important in describing the state of periodic motion, is not essential for determining the frequency (and the energy levels in the quantum context) directly, and therefore will not be considered here.

III. CLASSICAL WEAKLY RELATIVISTIC HARMONIC OSCILLATOR

The relativistic motion of the harmonic oscillator is governed by the Hamiltonian $H(x, p) = \sqrt{p^2c^2 + m^2c^4} + \frac{1}{2}kx^2$. The total mechanical energy of the relativistic oscillator will be referred to as E, and $\tilde{E} = E - mc^2$ is its mechanical energy in excess of its rest

mass energy. The dimensionless energy related parameter we will use is $\epsilon = \frac{\tilde{E}}{mc^2}$. We define the weak relativistic motion of the oscillator as one characterized by $\epsilon << 1$, and evaluate all dynamical variables up to the first order in ϵ. We approximate the Hamiltonian for the relativistic oscillator by $H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{kx^2}{2}$, retaining only the leading relativistic term in kinetic energy, and obtain the oscillator's approximate relativistic frequency using the two forms of the action variable discussed earlier. Such a study of the classical oscillator, with an approximation of the relativistic Hamiltonian, although unnecessary for frequency determination (at least three different series solutions exist for the frequency of a fully relativistic oscillator [15]), is a necessary prelude to the consideration of the corresponding quantum system within the framework of a non-relativistic quantum theory, to be discussed in the next section. Using T for the non-relativistic kinetic energy $\frac{p^2}{2m}$ the correctional relativistic term in H can be written as $-\frac{1}{2}T(\frac{T}{mc^2})$. The factor $\frac{T}{mc^2}$ is of order ϵ, and can be treated as a perturbative parameter in the non-relativistic Hamiltonian. This approximation of the Hamiltonian is good for studying motion in the weakly relativistic case. For consistency we should retain terms only up to order ϵ^1 in expressions for p_{CWR}, J_{CWR} and ω_{WR}.

We first consider the classical action variable in the $\oint p dx$ form. The classical orbit equation is $\frac{{p_{CWR}}^2}{2m} - \frac{{p_{CWR}}^4}{8m^3c^2} + \frac{kx^2}{2} = \tilde{E}$. Extending this into the complex-x plane the momentum is a branch of

$$p_{CWR}(x, \tilde{E}) = \sqrt{2}\, mc \sqrt{1 - \sqrt{1 - \frac{2}{mc^2}\left(\tilde{E} - \frac{1}{2}kx^2\right)}}. \tag{11}$$

Two turning points emerge at x_1 and x_2, identical in form to those in the fully relativistic case, given by $-x_1 = x_2 = \sqrt{\frac{2\tilde{E}}{k}}$. These are also two of the branch points of $p_{CWR}(x, E)$. There are two additional branch points, x_{3WR} and x_{4WR}, arising from the weak relativistic correction in the Hamiltonian, given by

$$-x_{3WR} = x_{4WR} = \sqrt{\frac{2\tilde{E}}{k}}\sqrt{1 - \frac{1}{2\epsilon}}. \tag{12}$$

For $\epsilon << \frac{1}{2}$ these branch points are on the imaginary axis with x_{4WR} farther from the origin than x_2. We choose one branch cut of p_{CWR} to connect x_1 and x_2 along the real axis, and two other cuts, each connecting x_{3WR} and x_{4WR} to $x = \infty$ along the imaginary axis. We choose the sign of $p_{CWR}(x, \tilde{E})$ to be positive below the cut joining x_1 and x_2. p_{CWR} is

analytic for $x_2 < |x| < x_{4WR}$ and its Laurent series in this annulus can be written as

$$p_{CWR}(x,E) = i\sqrt{mk}\sqrt{1+\epsilon}\; x\left[1-\left(\frac{x_2}{x}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{2}{1+\sqrt{1-(\frac{x}{x_{4WR}})^2}\sqrt{1-2\epsilon}}} = \sum_{j=-\infty}^{\infty} A_{Wj}x^{3-2j}. \tag{13}$$

Here the square roots can be expanded using binomial series. The action variable $J_{CWR}(\tilde{E})$, defined as $\frac{1}{2\pi}\oint_{C_{WR}} p_{CWR}(x,E)dx$, where C_{WR} is a counterclockwise rectangular contour that hugs the branch cut connecting x_1 and x_2, is evaluated up to order ϵ^1 by expanding the square roots in Eq. (13) and integrating using Cauchy's residue theorem. That yields

$$J_{CWR}(\tilde{E}) = \frac{1}{2\pi}(2\pi i)A_{W2} \approx i\;.\;i\sqrt{mk}\;\left(-\frac{\tilde{E}}{m{\omega_0}^2}\right)\;\left[1+\frac{3}{16}\frac{\tilde{E}}{mc^2}\right] \;=\; \frac{\tilde{E}}{\omega_0}\;\left(1+\frac{3}{16}\epsilon\right). \tag{14}$$

This differs from $\frac{\tilde{E}}{\omega_0}$, the action variable for the non-relativistic harmonic oscillator, with a correction of order ϵ. The angular frequency is obtained from

$$\frac{1}{\omega_{WR}} = \frac{\partial J_{CWR}}{\partial E} = \frac{1}{\omega_0}(1+\frac{3}{8}\epsilon), \tag{15}$$

which shows that the weak relativistic correction results in a fractional decrease of about $\frac{3}{8}\epsilon$ in the oscillator's frequency.

We now consider the second form of the action variable for the weakly relativistic case. The orbit equation can be solved for the coordinate to yield $x_{CWR}(p,\tilde{E}) = \sqrt{\frac{2}{k}}\;\left[\tilde{E}-\frac{p^2}{2m}+\frac{p^4}{8m^3c^2}\right]^{\frac{1}{2}}$. The two physical turning momenta of this oscillator, where the coordinate vanishes, are defined by

$$-\,p_{WR1} = p_{WR2} = \sqrt{2}mc\left[1-(1-2\epsilon)^{\frac{1}{2}}\right]^{\frac{1}{2}} \;\approx\; \sqrt{2m\tilde{E}}\left(1+\frac{\epsilon}{4}\right). \tag{16}$$

These are very nearly the non-relativistic turning momenta, given by $-p_1 = p_2 = \sqrt{2m\tilde{E}}$. They contain a weak relativistic correction of order ϵ. There are two additional branch points, $p_{WR3,4}$ of (weak) relativistic origin given by

$$-\,p_{WR3} = p_{WR4} = \sqrt{2}mc\left[1-(1-2\epsilon)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{17}$$

We write $x_{CWR}(p,\tilde{E})$ in the form

$$x_{CWR}(p,\tilde{E}) = \sum_{j=-\infty}^{\infty} A'_{Wj}p^{3-2j} = \frac{-i}{\sqrt{mk}}\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,(1-2\epsilon)^{\frac{1}{2}}\right]^{\frac{1}{2}} p\left[1-\left(\frac{p_{WR2}}{p}\right)^2\right]^{\frac{1}{2}}\left[1-\left(\frac{p}{p_{WR4}}\right)^2\right]^{\frac{1}{2}}.$$

Using this we extend $x_{CWR}(p,\tilde{E})$ into the complex-p plane. We choose one branch cut of $x_{CWR}(p,\tilde{E})$ from p_{1WR} to p_{2WR} along the real axis (the function is chosen positive just below this cut), and two other cuts, each of which joins p_{3WR} and p_{4WR} to $p=\infty$ along the real axis. The alternate definition of $J_{CWR}(\tilde{E})$ is

$$J_{CWR}(\tilde{E}) = -\frac{1}{2\pi}\oint_{C'_{WR}} x_{CWR}(p,\tilde{E})dp, \tag{18}$$

where the counterclockwise rectangular contour C'_{WR} wraps around the branch cut connecting p_{1WR} and p_{2WR}. For $p_{WR2} < |p| < p_{WR4}$ we expand x_{CWR} in a Laurent series and evaluate J_{CWR} to obtain

$$J_{CWR} = \frac{\tilde{E}}{\omega_0}\left[\frac{2}{1+\sqrt{1-2\epsilon}}\right]^{\frac{1}{2}}\left[1-\frac{1}{8}\left(\frac{p_{WR2}}{p_{WR4}}\right)^2\ldots\right]. \tag{19}$$

This representation of the action variable contains a series in powers of $(p_{WR2}/p_{WR4})^2 = \left(\frac{1-\sqrt{1-2\epsilon}}{1+\sqrt{1-2\epsilon}}\right) \approx \frac{\epsilon}{2}$. To order ϵ it is $\frac{\tilde{E}}{\omega_0}[1+\frac{3}{16}\epsilon]$, consistent with (14). The angular frequency is found from

$$\frac{1}{\omega_{WR}} = \frac{dJ_{CWR}}{d\tilde{E}} = \frac{1}{\omega_0}\frac{d}{d\epsilon}\left[\epsilon\left\{\frac{2}{1+\sqrt{1-2\epsilon}}\right\}^{\frac{1}{2}}\left\{1-\frac{1}{8}\left(\frac{1-\sqrt{1-2\epsilon}}{1+\sqrt{1-2\epsilon}}\right)\ldots\right\}\right] \approx \frac{1}{\omega_0}(1+\frac{3}{8}\epsilon), \tag{20}$$

which agrees with the result in Eq. (15). Further both results agree, up to order ϵ^1, with the expressions for the angular frequency obtained by using the fully relativistic Hamiltonian [15].

IV. HARMONIC OSCILLATOR IN QUANTUM HAMILTON-JACOBI THEORY

A. Quantum Action Variable

We summarize here the formalism of Hamilton-Jacobi quantum mechanics[9,10], equivalent to other better known ones, and apply it to the special case of 1-D one particle systems with Hamiltonians of the form $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})$. The measurable values of the observables $\hat{H}, \hat{p}$ and $\hat{x}$ are their eigenvalues E, p and x respectively. The equations of quantum canonical transformation are written in terms of the eigenvalues and functions of eigenvalues of these observables. Using the quantum characteristic function $W(x,P)$ these transformation equations, equivalent to the classical ones in Eq. (3), are

$$p = \frac{\partial W(x,P)}{\partial x}, \qquad X = \frac{\partial W(x,P)}{\partial P}. \tag{21}$$

The *quantum Hamilton-Jacobi equation*, for systems with time independent Hamiltonians, is

$$-i\hbar\frac{\partial^2 W(x,E(P))}{\partial x^2}+\left(\frac{\partial W(x,E(P))}{\partial x}\right)^2 = E(P)-V(x). \tag{22}$$

This can be either postulated or derived from Schrdinger's equation for the state $\psi(x,t)$, which is written in the form $e^{\frac{i}{\hbar}S(x,t)}$, where $S(x,t)$ is Hamilton's principal function in the quantum context. For systems with time independent Hamiltonians, W and S are related through $S(x,P,t)=W(x,P)-Et$. Physical boundary conditions have to be imposed on $W(x,E(P))$ to complete its definition. We note that this equation resembles the classical Hamilton-Jacobi equation (5), except for the additional term involving $\hbar$, and reduces to it in the limit $\hbar \rightarrow 0$. The dynamics described by this equation is non-relativistic and is equivalent in all respects to the Schrdinger formalism of quantum mechanics. We can use such an equation only in a perturbative sense to treat systems with weak relativistic terms in their Hamiltonian.

$p(x,E)=\frac{\partial W(x,P(E))}{\partial x}$ is the quantum analog of the classical momentum function $p_C(x,E)$ and, following Ref. 10, will be referred to as the *quantum momentum function*. Using this definition of $p(x,E)$ in Eq. (22) we obtain

$$-i\hbar\frac{\partial p(x,E)}{\partial x}+p^2(x,E) \;=\; 2m[E-V(x)] \;=\; {p_c}^2(x,E). \tag{23}$$

We note that the square of this quantum momentum function reduces to the square of $\pm p_c(x,E)$, the classical momentum function in the phase space orbit equation, in the limit $\hbar \rightarrow 0$. The physical boundary condition on $p(x,E)$ is that for all x

$$\lim_{\hbar\rightarrow 0} p(x,E) \;\;=\;\; +p_c(x,E). \tag{24}$$

Eq. (23), along with (24), defines $p(x,E)$ which we will use in the construction of the quantum action variable J. Equation (23) is the Ricatti (nonlinear) form of the Schrdinger (linear) differential equation. Extending the wave function $\psi_E(x,t) = \phi(x,E)e^{-\frac{iEt}{\hbar}}$ into the complex x plane it is easily shown that the nodes of $\phi(x,E)$ are also the poles of $p(x,E)$. Using oscillation theorems for the linear differential equation it can be shown that (i) $p(x,E)$ has simple poles of residue $-i\hbar$ on the real axis between the two turning points independent of the system's energy, and (ii) there are fixed poles whose locations and residues are determined by the specific nature of the potential.[10,13]. The number of poles between the turning points counts the level of excitation of this quantum system.

Further development of this quantum formalism for periodic systems, where action-angle variables can be employed, requires the specification of a new momentum P, which is the *quantum action variable* J. Following the definition of the classical action variable it is defined as the contour integral in the complex x plane,

$$J(E) = \frac{1}{2\pi} \oint_C p(x, E) dx, \tag{25}$$

with the counterclockwise rectangular contour C tightly wrapping the real axis between the two physical turning points of the *classical* momentum function $p_C(x, E) = \sqrt{E - V(x)}$. Since C encloses only a finite number of poles of $p(x, E)$, deforming C to encircle the poles and integrating using the residue theorem leads to the discretization of J:

$$J = \frac{1}{2\pi}(2\pi i)\{n(-i\hbar)\} \quad = \quad n\hbar. \tag{26}$$

This is in contrast to the classical action variable which assumes continuous values. At first glance, Eq. (26) seems to be the Wilson-Sommerfeld quantization condition or the JWKB quantum rule, but it is not. The action variable (or the Sommerfeld phase integral) used in those conditions is the *classical* action variable; the one used here is the quantum action variable whose definition is based on the quantum momentum function $p(x, E)$ which is a solution of Eq. (23) whose basis is in quantum mechanics. Finally, By deforming C outward and integrating we capture the energy dependence of the quantum action variable and obtain $J(E)$. Thus $J(E) = n\hbar$, and inverting this, we obtain the system's energy eigenvalues, $E = E(J = n\hbar)$.

As will be shown later, the integral in Eq. (25) can be performed without obtaining a solution of Eq. (23) all over the complex x plane. The quantum energy eigenvalues of a system that is classically periodic can thus be obtained more simply by using the discretizatized nature of the quantum action variable than by imposing boundary conditions on the wave function. It can be shown that the poles of $p(x, E)$ on the real axis between the physical turning points coalesce in the classical limit, and form the branch cut of $p_c(x, E)$. The mechanism of this pole coalescing is demonstrated in Ref. 10. The residue of a pole of $p(x, \tilde{E})$ between the turning points is proportional to $\hbar$, and so is the spacing between neighboring poles. In the limit $\hbar \to 0$, for a fixed energy of the oscillator, the real axis between the turning points is riddled with poles, with the quantum momentum function having opposite signs on the sides of the pole above and below the real axis, giving rise to the branch cut of $p_C(x, E)$.

We will construct two such equivalent forms of the action variable J for the simple harmonic oscillator in the quantum context and demonstrate that both forms of J are discrete. Since the energy is a function of the action variable, it too is discrete rather than continuous. It is important to recognize that (1) this quantum action variable differs from the classical action variable employed by Sommerfeld and others for quantization, (2) the quantum action variable naturally assumes values that are integral multiples of $\hbar$ and (3) the energy eigenvalues obtained by evaluating the action variable are exact. By making appropriate approximations of this action variable as a function of energy we obtain approximate energy eigenvalues.

B. $\oint p dx$ form of quantum action variable

The quantum action variable for the harmonic oscillator is defined through Eq. (25) where the rectangular counterclockwise contour C closely wraps around the real axis between the turning points x_1 and x_2. For evaluating the integral we deform C outward into the circular contour γ centered at the origin, and use the Laurent series for $p(x, E)$ in an origin centered annulus that includes γ. The boundary condition on $p(x, E)$ indicates the form of the Laurent series that it has in this annulus. The Laurent series for the classical momentum function is

$$p_C(x, E) = \sqrt{E - \frac{1}{2}kx^2} = i\sqrt{mk}x \left[1 - \left(\frac{x_2}{x}\right)^2\right]^{\frac{1}{2}} = \sum_{j=1}^{\infty} a_j x^{3-2j}. \tag{27}$$

We choose the form for the series for the quantum momentum function that is similar to that of the series for $p_C(x, E)$ and write it as

$$p(x, E) = \sum_{j=1}^{\infty} b_j x^{3-2j}. \tag{28}$$

Substituting this in Eq. (23) and equating coefficients of like powers of x we get $b_1 = \pm i\sqrt{mk}$. We choose the + sign here since $a_1 = +i\sqrt{mk}$; the boundary condition on $p(x, E)$ requires $b_j \to a_j$ in the classical limit. The next coefficient is $b_2 = -i\sqrt{\frac{m}{k}}E + i\frac{\hbar}{2}$. Using the residue theorem we get

$$J(E) = 2\pi i \, (\frac{1}{2\pi}) \, b_2 = E\sqrt{\frac{m}{k}} - \frac{\hbar}{2} \tag{29}$$

This, along with the discretization condition in Eq. (26), yields $E = (n + \frac{1}{2})\hbar$, the energy eigenvalues of the harmonic oscillator.

C. Equivalent quantum Hamilton-Jacobi equation

We present here a new version of the quantum Hamilton-Jacobi equation that is equivalent to the one in Eq. (22) using an alternate quantum canonical transformation scheme. While the dynamical equation can be postulated for the quantum generating function $\tilde{S}(p, X, t)$ that generates the transformation

$$x = -\frac{\partial \tilde{S}(p, X, t)}{\partial p}, \quad P = -\frac{\partial \tilde{S}(p, X, t)}{\partial X},$$

and show its equivalence to the Schrdinger equation, we choose the reverse route and derive it from the latter. Starting with the Schrdinger equation

$$\left[\frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})\right] |\phi> = i\hbar \frac{\partial |\phi>}{\partial t} \tag{30}$$

and using the momentum representation where $\hat{x} \rightarrow i\hbar \frac{\partial}{\partial p}$, $\hat{p} \rightarrow p$ and $|\phi> \rightarrow \phi(p, E, t)$, we get

$$\left[\frac{p^2}{2m} + V\left(i\hbar \frac{\partial}{\partial p}\right)\right] \phi = i\hbar \frac{\partial \phi}{\partial t}. \tag{31}$$

Introducing the alternate quantum Hamilton's principle function $\tilde{S}(p, X, t)$ through $\phi(p, E(X), t) = e^{\frac{i}{\hbar}\tilde{S}(p,X,t)}$ and defining G from

$$V(i\hbar \frac{\partial}{\partial p})\, \phi(p, E, t) = G\left(\hbar, \tilde{S}_p, \tilde{S}_{pp}, \ldots\right)\, e^{\frac{i}{\hbar}\tilde{S}(p,X,t)} \tag{32}$$

we obtain

$$\frac{p^2}{2m} + G(\hbar, \tilde{S}_p, \tilde{S}_{pp}, \ldots) = -\tilde{S}_t. \tag{33}$$

Here suffixes denote partial differentiation. We define the alternate quantum characteristic function $\tilde{W}(p, X)$ through the relation $\tilde{S}(p, X, t) = \tilde{W}(p, X) - Et$ and use it in Eq. (33) to get

$$\frac{p^2}{2m} + G(\hbar, \tilde{W}_p, \tilde{W}_{pp}, \ldots) = E. \tag{34}$$

This is the quantum Hamilton-Jacobi equation for $\tilde{W}(p, X)$ which generates a canonical transformation from (x, p) to (X, P). Use of the quantum canonical transformation equation, $x = -\frac{\partial \tilde{W}}{\partial p}$, in Eq. (34) leads to

$$\frac{p^2}{2m} + G\left(\hbar, x(p, E), \frac{\partial x(p, E)}{\partial p}, \ldots\right) = E. \tag{35}$$

This is the equivalent of Eq. (23) under this alternate canonical transformation scheme. The physical boundary condition on $x(p, E)$ is that in the limit $\hbar \to 0$, it should reduce to the classical coordinate function, $x_C(p, E)$. Specializing to the case of the simple harmonic oscillator we get

$$G = \frac{i}{\hbar}\frac{\partial x(p, E)}{\partial p} - \frac{1}{\hbar^2}x^2(p, E). \tag{36}$$

Substituting this in Eq. (35) we obtain the quantum equivalent of the classical orbit equation:

$$i\hbar\frac{\partial x}{\partial p} + x^2 = \frac{2}{k}\left[E - \frac{p^2}{2m}\right]. \tag{37}$$

In the limit $\hbar \to 0$ this equation becomes, for real values of x and p, the classical orbit equation that constrains the harmonic oscillator to an elliptical path in phase space.

We choose that quantum canonical transformation which makes the quantum action variable J the new coordinate. It is defined as the contour integral

$$J = -\frac{1}{2\pi}\oint_{C'} x(p, E)dp \tag{38}$$

in the complex-p plane, with the clockwise rectangular contour C' enclosing the real axis between the two turning momenta p_1 and p_2. A comparison of Eq. (37) with Eq. (23) shows that, as in the case of the quantum momentum function $p(x, E)$, the quantum coordinate function $x(p, E)$ has a finite number of poles on the real axis between p_1 and p_2, each of residue $+i\hbar$. In the classical limit these poles coalesce to form the branch cut of $x_C(p, E)$. The steps for the determination of energy eigenvalues here parallels that described in the previous section. We evaluate J in two ways, one by outward deformation of C' into the origin centered circular contour γ' and integrating to obtain $J = J(E)$, and, two, through inward deformation, encircling the poles and integrating, to get $J = n\hbar$.

For $|p| > p_2$, the form of the Laurent series for the classical momentum function $x_C(p, \tilde{E})$ is

$$x_C(p, \tilde{E}) = \sqrt{\frac{2}{k}\left[E - \frac{p^2}{2m}\right]} = \frac{-i}{\sqrt{mk}}p\left[1 - \left(\frac{p_2}{p}\right)^2\right]^{\frac{1}{2}} = \sum_{j=-\infty}^{\infty} a'_j p^{3-2j}.$$

We use a similar series form for the quantum coordinate function $x(p, E)$ valid for this region of the complex plane:

$$x(p, E) = \sum_{j=1}^{\infty} b'_j p^{3-2j}. \tag{39}$$

Substituting this in Eq. (35), and imposing the physical boundary condition on $x(p,E)$, we obtain

$$b_1' = \frac{-i}{\sqrt{mk}},\ b_2' = -i\left(\frac{\hbar}{2} - \frac{E}{\omega_0}\right) \tag{40}$$

Expanding the contour C' outward to γ' and evaluating J we get

$$J = -\frac{1}{2\pi}(2\pi i)\, b_2' = \frac{E}{\omega_0} - \frac{\hbar}{2}. \tag{41}$$

Shrinking the contour inward to encircle the poles on the real axis between p_1 and p_2 and integrating yields $J = n\hbar$. So, $\frac{E}{\omega_0} - \frac{\hbar}{2} = n\hbar$, and inverting this relation we obtain the energy eigenvalues of the simple harmonic oscillator, $E = (n + \frac{1}{2})\hbar\omega_0$.

This alternate form of the quantum action variable has the same physical content as the previous form. While the first form is easier to evaluate for systems whose potential energy functions are not necessarily quadratic in x, the second form is better suited for the weakly relativistic oscillator, as will be shown in the next section.

V. QUANTUM MECHANICS OF WEAKLY RELATIVISTIC OSCILLATOR

A. Integral $\oint p dx$ form of action variable

The Hamiltonian for the weakly relativistic oscillator is $\frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3c^2} + \frac{1}{2}k\hat{x}^2$. Using this in the quantum Hamilton-Jacobi equation for $W(x,P)$ (we omit the subscript WR for W), which is obtained from Schrdinger's equation following the steps outlined in Section IV.A, we get

$$\tilde{E} = \frac{1}{2}kx^2 + \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2 W}{\partial x^2} - \frac{1}{8m^3c^2}\left[\left(\frac{\partial W}{\partial x}\right)^4 - 6i\hbar\left(\frac{\partial W}{\partial x}\right)^2\frac{\partial^2 W}{\partial x^2} - \hbar^2\left\{4\frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x^3} + 3\left(\frac{\partial^2 W}{\partial x^2}\right)^2\right\} + i\hbar^3\frac{\partial^4 W}{\partial x^4}\right] \tag{42}$$

Using the quantum canonical transformation equation $p = \partial W/\partial x$ in Eq. (42) results in

$$\tilde{E} - \left[\frac{p^2}{2m} + \frac{1}{2}kx^2 - \frac{p^4}{8m^3c^2}\right] = -\frac{i\hbar}{2m}\frac{\partial p}{\partial x} + \frac{\hbar}{2m^3c^2}\left[\frac{3}{2}ip^2\frac{\partial p}{\partial x} + \hbar\left\{\frac{3}{4}\left(\frac{\partial p}{\partial x}\right)^2 + p\frac{\partial^2 p}{\partial x^2}\right\} - \frac{i\hbar^2}{4}\frac{\partial^3 p}{\partial x^3}\right] \tag{43}$$

Eq. (43) shows the presence of terms proportional to $\hbar$ or its higher powers, which are absent in the corresponding classical orbit equation. Further, in the non-relativistic limit it reduces to Eq. (23).

Using the solution p_{WR} of Eq. (43) satisfying the physical boundary condition (i.e., has the correct classical limit) we define the quantum action variable as

$$J_{WR} = \frac{1}{2\pi} \oint_{C_{WR}} p_{WR}(x, \tilde{E}) dx, \tag{44}$$

using the same contour C_{WR} as in the classical case. Following the form of $p_{CWR}(x, \tilde{E})$ in Eq. (13) the Laurent series for $p_{WR}(x, \tilde{E})$ in the annulus $x_2 < |x| < x_{4WR}$, correct to order ϵ^1, is of the form

$$p_{WR}(x, \tilde{E}) = \left[\sum_{j=1}^{\infty} b_j x^{3-2j} \right] \left[1 + \frac{\epsilon}{4} \left\{ B_0 - B_1 \left(\frac{x}{x_2} \right)^2 \right\} \right]. \tag{45}$$

The non-relativistic quantum momentum function in Eq. (28) has been modified here by the addition of a relativistic term of order ϵ. We have chosen the form of this series such that (i) the relativistic modification is of the first order in ϵ and in the limit $\epsilon \to 0$ we get $p_{WR}(x, \tilde{E}) \to p(x, \tilde{E})$, and (ii) in the limit $\hbar \to 0$ we obtain $p_{WR}(x, \tilde{E}) \to p_{CWR}(x, \tilde{E})$. The coefficients b_j are known from the non-relativistic quantum case previously considered (see Section IV.B). Substituting this series form of the solution in Eq. (43) we obtain the coefficients B_0 and B_1:

$$B_0 = 1, \quad B_1 = 1 + \frac{7\hbar\omega_0}{4\tilde{E}}. \tag{46}$$

The coefficient of x^{-1} in $p_{WR}(x, \tilde{E})$ is $im\omega_0 \; x_2^2 \; (b_1 B_0 - b_0 B_1)$. Evaluating J by deforming the contour C_{WR} outward to the origin centered circular contour γ_{WR} previously considered, we get

$$\begin{aligned} J_{WR} &= 2\pi i \left(\frac{1}{2\pi} \right) \; im\omega_0 \; x_2^2 \; (b_1 B_0 - b_0 B_1) \\ &= \frac{\tilde{E}}{\omega_0} \left[1 + \epsilon \left\{ \frac{3}{16} + \frac{7}{16} \left(\frac{\hbar\omega_0}{\tilde{E}} \right) - \frac{17}{64} \left(\frac{\hbar\omega_0}{\tilde{E}} \right)^2 \right\} \right] - \frac{\hbar}{2} \end{aligned} \tag{47}$$

A comparison with Eq. (29) shows the relativistic corrections present in this quantum action variable up to order ϵ. Solving Eq. (47) for $\tilde{E}$, we get

$$\tilde{E} \; = \; \tilde{E}(J_{WR} = n\hbar) \; = \; \left[\left(n + \frac{1}{2} \right) - \frac{3}{16} \left\{ \left(n + \frac{5}{3} \right)^2 - \frac{25}{9} \right\} \frac{\hbar\omega_0}{mc^2} \right] \; \hbar\omega_0 \tag{48}$$

This gives energy eigenvalues with a first order relativistic correction that is proportional to the dimensionless energy parameter $\frac{\hbar\omega_0}{mc^2}$, which measures the level separation in the non-relativistic case. The energy levels in the weak relativistic case are lower than in the non-relativistic case, and the lowering is predominantly quadratic in the quantum number n. The seperation between energy levels is proportional to n.

B. $-\oint x dp$ form of action variable

We now apply the formalism developed in Section IV.C to construct the quantum action variable in its alternate form. Using the Hamiltonian for the weakly relativistic oscillator in the quantum Hamilton-Jacobi equation for the alternate characteristic function $\tilde{W}(p, X)$ we get

$$-i\hbar\frac{\partial^2\tilde{W}}{\partial p^2} + \left(\frac{\partial\tilde{W}}{\partial p}\right)^2 = \frac{2}{k}\left[\tilde{E} - \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2}\right)\right]. \tag{49}$$

In comparison to the quantum Hamilton-Jacobi equation (42), this alternate form is simpler as it has a single quantum term, and is to be preferred in the study of this oscillator. Introducing the quantum canonical transformation equation $x = -\partial\tilde{W}/\partial p$ in Eq. (49), we obtain

$$i\hbar\frac{\partial x}{\partial p} + x^2 = \frac{2}{k}\left[\tilde{E} - \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2}\right)\right]. \tag{50}$$

We impose the physical boundary condition that in the limit $\hbar \to 0$, the quantum coordinate function $x(p, E)$ should reduce to the classical coordinate function $x_C(p, E)$ for all p. This equation has the same structure as its non-relativistic counterpart, Eq. (37).

Following the classical case, we define the quantum action variable as $J_{WR} = -\frac{1}{2\pi}\oint_{C'_{WR}} x_{WR}(p, E)dp$, where x_{WR} is the solution of Eq. (50) that satisfies the physical boundary condition. The construction of this form of the quantum action variable and the derivation of energy eigenvalues in this case is identical in all relevant details to the problem of the the non-relativistic anharmonic oscillator with a quartic potential energy term δx^4, and with J in the $\oint p(x, E)dx$ form, shown in the Appendix. A comparison with the latter problem indicates that we can find the action variable for the weakly relativistic oscillator by making the replacements $x \to -p, p \to -x, \frac{\delta}{k^2} \to \frac{-1}{8mc^2}$ in Eq. (57) and get

$$J_{WR}(\tilde{E}) = \frac{\tilde{E}}{\omega_0} - \frac{\hbar}{2} + \frac{3\hbar}{64}\left\{1 + \frac{4\tilde{E}^2}{(\hbar\omega_0)^2}\right\}\left(\frac{\hbar\omega_0}{mc^2}\right), \tag{51}$$

correct to the first order in $\frac{\hbar\omega_0}{mc^2}$. Solving for $\tilde{E}$ in this order using $J_{WR}(E) = n\hbar$, we get

$$\tilde{E} = \left[\left(n + \frac{1}{2}\right) - \frac{3}{16}\left(\frac{\hbar\omega_0}{mc^2}\right)\left\{\left(n + \frac{1}{2}\right)^2 + 4\right\}\right]\hbar\omega_0. \tag{52}$$

The above energy eigenvalues obtained by approximating the quantum coordinate function, while not identical to those in Eq. (48), display the same features that were observed earlier. Both forms of the quantum action variable indicate that the lowering of energy levels due to the weak relativistic correction is approximately proportional to n^2.

VI. OSCILLATOR FREQUENCIES AND ENERGY LEVELS FROM DIFFERENT SCHEMES: A COMPARISON

Table 1 summarizes the expressions for the classical action variable of the relativistic oscillator under four different schemes, two of which we have considered here. The first two rows show two equivalent series representations for J_C for the fully relativistic oscillator with no approximation[15], and the next two rows the corresponding representations obtained by using the weak relativistic approximation for the kinetic energy. To the first order in ϵ they all yield the same expression. Relativistic dynamics lowers the oscillator's frequency due to time dilation, and the fraction by which the frequency is lowered is $\frac{3}{8}\epsilon$ for this weakly relativistic oscillator. The quantum action variable for the weakly relativistic oscillator under the two alternate action variable schemes we considered is shown in Table 2. The energy eigenvalues obtained from these schemes and those from two other approximation schemes are also shown in this table. The semiclassical JWKB approximation involves discretization of the *classical* action variable $J_{CWR}(\tilde{E})$ in Eq. (14) for the weakly relativistic oscillator:

$$J_{CWR}(\tilde{E}) \;=\; \frac{\tilde{E}}{\omega_0}\left[1 + \frac{3}{16}\epsilon\right] \;=\; (n + \frac{1}{2})\,\hbar.$$

In the Rayleigh-Schrdinger perturbation scheme the shifts in energy eigenvalues from their unperturbed values, to the first order in ϵ, are $\Delta\tilde{E}_n =< \phi_n|\hat{H}_\epsilon|\phi_n >$, treating $\hat{H}_\epsilon = -\frac{p^4}{8m^3c^2}$ as the perturbation term in the Hamiltonian whose zeroth order form is $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$. In all four schemes the calculated weak relativistic shift in the energy eigenvalues from thier non-relativistic values is approximately $-\frac{3}{16}\left(\frac{\hbar\omega_0}{mc^2}\right)n^2$. We find that the energy level spacing is

$$E_{n+1} - E_n \;\approx\; \hbar\omega_0\left[1 - \frac{3}{8}n\,\frac{\hbar\omega_0}{mc^2}\right] \;\approx\; \hbar\omega_0\left[1 - \frac{3}{8}\,\epsilon\right] \;=\; \hbar\omega_{WR}.$$

The angular frequency ω_{WR}, unlike in the non-relativistic case, is energy dependent. Thus the spacing between the quantum energy levels in a range which conform to this approximation scheme is proportional to the classical oscillator's weak relativistic angular frequency for that energy range. Viewed differently, the weak relativistic correction produces a fractional shift of $\frac{3}{8}\epsilon$ in the angular frequency of the classical oscillator. There is an identical fractional shift in the energy level separation of the corresponding quantum oscillator.

VII. QUANTUM - CLASSICAL AND RELATIVISTIC - NON-RELATIVISTIC CORRESPONDENCE

The physically correct theory of matter should be both quantum in nature and meet the principle of relativity. The Klein-Gordon and Dirac equations represent physical theories that meet these two physical requirements. The description of the weakly relativistic oscillator considered here meets the first principle. While it does not meet the second principle it does incorporate a weak relativistic dynamical correction. It is a useful treatment of an "energetic" oscillator as it begins to approach the relativistic regime. We notice the operation of two correspondence principles in the dynamics of this physical system. One is the reduction of the relativistic model to its appropriate, well established non-relativistic form. The other is the similar reduction of the quantum model to its corresponding classical form. Both the weak relativistic quantum Hamilton-Jacobi equations (42) and (49) reduce to their weak relativistic *classical* counterparts in the limit $\hbar \to 0$, and these equations, in turn, assume their classical non-relativistic form for $\epsilon << 1$. Secondly, the poles of the quantum momentum function $p_{WR}(x, \tilde{E})$ in the complex-x plane, and those of the quantum coordinate function $x_{WR}(p, \tilde{E})$ in the complex-p plane coalesce to form the branch cuts of classical functions $p_{CWR}(x, \tilde{E})$ and $x_{CWR}(p, \tilde{E})$ respectively. Thirdly, the expressions for $J_{WR}(\tilde{E})$ in the quantum case reduce to those of $J_{CWR}(\tilde{E})$ in the classical non-relativistic limit. Fourthly, the weakly relativistic quantum action variable $J_{WR}(\tilde{E})$ reduces to the non-relativistic quantum action variable $J(\tilde{E})$ for $\epsilon << 1$. This formalism of quantum mechanics, applied to the harmonic oscillator, thus shows in a unified framework the critical role played by the two "small" parameters, $\hbar$ and ϵ, in the emergence of non-relativistic classical theory from non-relativistic quantum theory and a non-relativistic quantum model from a weakly relativistic quantum model.

VIII. CONCLUSION

The use of action-angle variables in the study of classical periodic systems has a parallel in quantum mechanics. Using the quantum action variable the exact energy eigenvalues of a bound system that is classically periodic can be obtained without a detailed solution of the dynamical equations. We have demonstrated the construction of two forms of the quantum action variable, and applied it to the harmonic oscillator to obtain its energy levels. We have extended the use of this formalism to a weakly relativistic harmonic oscillator. The classical frequency of such an oscillator is lowered from its non-relativistic value by the fraction $\frac{3}{8}\epsilon$. There is an identical fractional shift in the energy level separation of the corresponding quantum harmonic oscillator due to the leading order relativistic correction. While the problem must be properly addressed by a relativistic quantum theory we have shown how this system may be studied in a unified manner within a non-relativistic quantum Hamilton-Jacobi theory.

IX. APPENDIX: ANHARMONIC OSCILLATOR WITH QUARTIC TERM

We consider an approximation scheme for obtaining the energy eigenvalues for a quartic anharmonic oscillator described by the potential $V(x) = \frac{1}{2}kx^2 + \delta x^4$, with $\delta > 0$, using the quantum action variable. The $\delta < 0$ case can be treated in a similar manner. The term δx^4 is considered "small" for $|x| < {x_2}^{(0)}$ in comparison to the dominant term $\frac{1}{2}kx^2$. Dynamical variables specific to this oscillator have the suffix AHO. We denote the physical turning points for the $\delta = 0$ (or the simple harmonic oscillator) case by $-x_1^{(0)} = x_2^{(0)} = \sqrt{\frac{2E}{k}}$. x_1 and x_2 are the physical turning points in the presence of the quartic potential term, and are approximately of magnitude $x_2^{(0)}\left(1 - \frac{2E}{k^2}\delta\right)$. For $\delta > 0$ they are closer to each other than those in the harmonic oscillator case. We seek the energy eigenvalues of this anharmonic oscillator correct to order δ^1. The classical momentum function $p_{CAHO}(x, E)$ has two other (unphysical) turning points, approximately given by $-x_3 = x_4 \approx i\sqrt{\frac{k}{2\delta}}\left(1 + \frac{2E\delta}{k^2}\right)$. We write the classical momentum function in the form

$$p_{CAHO}(x, E) = i\sqrt{mk}\; x \left[1 - \left(\frac{x_2^{(0)}}{x}\right)^2\right]^{\frac{1}{2}} \left[\frac{1 + \frac{\frac{2\delta}{k}x^2}{1 - \left(\frac{x_2^{(0)}}{x}\right)^2}}{}\right]^{\frac{1}{2}}. \tag{53}$$

To order δ^1 this has, for $x_2^{(0)} < |x| < |x_4|$, the series representation

$$p_{CAHO}(x,E) \approx i\sqrt{mk}\, x \ \left[\sum_{j=0}^{\infty} c_j \left(\frac{x_2^{(0)}}{x}\right)^{2j}\right] \ \left[1 + \frac{\delta}{k}x^2 \ \left\{\sum_{l=0}^{\infty}\left(\frac{x_2^{(0)}}{x}\right)^{2l}\right\}\right]. \qquad (54)$$

Here c_j are the coefficients in the binomial expansion of $\sqrt{1-u}$ in powers of u. We use a similar structure for the Laurent series, for the anharmonic oscillator's quantum momentum function $p_{AHO}(x,E)$, good for $x_2 < |x| < |x_4|$ and correct to order δ^1:

$$p_{AHO}(x,E) \approx i\sqrt{mk}\, x \ \left[\sum_{j=0}^{\infty} b_j \left(\frac{x_2^{(0)}}{x}\right)^{2j}\right] \ \left[1 + \frac{\delta}{k}x^2 \ \left\{\sum_{l=0}^{\infty} D_l \ \left(\frac{x_2^{(0)}}{x}\right)^{2l}\right\}\right] \qquad (55)$$

The coefficients b_j are known from our solution of the quantum harmonic oscillator problem (Section IV.B). We need to solve for the coefficients D_l using Eq. (23). The quantum action variable is defined as

$$J_{AHO}(E) = \frac{1}{2\pi}\oint_{C_{AHO}} p_{AHO}(x,E)dx, \qquad (56)$$

where the counterclockwise rectangular contour C_{AHO} wraps around the real axis between the turning points x_1 and x_2 (or $x_1^{(0)}$ and $x_2^{(0)}$ for the $\delta < 0$ case). The coefficient of x^{-1} in the series in Eq. (55) is $im\omega_0 {x_2^{(0)}}^2 \ \left[b_1 + \frac{\delta}{k}{x_2^{(0)}}^2 \ (b_2 D_0 + b_0 D_2 + b_1 D_1)\right]$.

We define the dimensionless parameter $\lambda = \frac{\hbar\omega_0}{4E}$, which is "small" for large values of the oscillator's energy. Introducing the above series for $p_{AHO}(x,E)$ in Eq. (55) and solving for the D-coefficients, we get $D_0 = 1,\ D_1 = 1+\lambda,\ D_2 = 1 - \frac{3}{2}\lambda + 2\lambda^2$.

Evaluating J_{AHO} by the outward deformation of the contour C_{AHO} into the circular counterclockwise contour γ_{AHO} we obtain

$$J_{AHO} = \hbar\left[-\frac{1}{2} + \frac{1}{4\lambda} - \frac{3\delta\hbar}{32m^2\omega_0^3}\ \left(4 + \frac{1}{\lambda^2}\right)\right]. \qquad (57)$$

Solving for E in $J_{AHO}(E) = n\hbar$ (obtained from inwardly deforming C_{AHO} to encircle the poles and integrating) we get the energy eigenvalues of the quartic anharmonic oscillator, to order δ^1:

$$E = \hbar\omega_0 \ \left[\left\{n + \frac{1}{2} + \frac{3}{8}\left(\frac{\delta}{k^2}\right)\ \hbar\omega_0\right\} + \frac{3}{2}\left(\frac{\delta}{k^2}\right)\ \hbar\omega_0 \ \left\{n + \frac{1}{2} + \frac{3}{8}\left(\frac{\delta}{k^2}\right)\ \hbar\omega_0\right\}^2\right] \qquad (58)$$

For $\delta > 0$, the quartic perturbation shifts the oscillator's energy levels higher (and lower for the $\delta < 0$ case), with the spacing between the energy levels increasing linearly with n.

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[1] C.E. Delaunay, " Mémoire sur une Méthode nouvelle pour la détermination du mouvement de la Lune ", (Académie des Sciences, Paris 1846).

[2] M. Jammer, "The Conceptual Development of Quantum Mechanics," (McGraw-Hill, New York 1966).

[3] K. Schwarzschild, "Zur Quanten-hypothese," Berliner Berichte 548-568 (1916).

[4] W. Wilson, "The quantum theory of radiation and line spectra", Philosophical Magazine textbf29, 795-802 (1915).

[5] A. Sommerfeld, "Zur Theorie der Balmerschen Serie," Mnchener Berichte, 425-458 (1915); "Die Fienstruktur der wasserstoff- und wasserstoffhnlichen Linien," Mnchener Berichte, 459-500 (1915); "Zur Quanten-theorie der Spektrallinien," Annalen der Physik **51** 1-94, 125-167 (1916).

[6] J. Ishiwara, "Die universelle Bedeutung des Wirkungsquantums," Tokyo Sugaku Buturigakkawi Kizi **8**, 106-116 (1915).

[7] M.J. Klein, "The origins of Ehrenfest's adiabatic principle," Proceedings of the Xth International Congress of History of Science, Cornell University, 1962 801-804 (Hermann, Paris, 1964).

[8] P. Ehrenfest, "Adiabatische Transformationen in der Quantentheorie und ihre Behandlung durch Niels Bohr," Die Naturwissenschaften **11** 543-550 (1923).

[9] R. A. Leacock and M. J. Padgett,"Hamilton-Jacobi Theory and the Quantum Action Variable," Phys. Rev. Lett. **50,** 3-6 (1983).

[10] R. A. Leacock and M. J. Padgett,"Hamilton-Jacobi/action-angle quantum mechanics," Phys. Rev. **D28**, 2491-2502 (1983).

[11] J.-H. Kim and H.-W.Lee,"Canonical transformations and the Hamilton-Jacobi theory in quantum mechanics," Can. J. Phys. **65**, 1183-1187 (1997).

[12] A. Nanayakkara,"Transformations in quantum action variable theory," J. Phys. A: Math. Gen. **23**, 2055-2064 (1990).

[13] R.S.Bhalla, A.K.Kapoor and P.K.Panigrahi,"Quantum Hamilton-Jacobi formalism and the bound state spectra," Am. J. Phys. **77**, 411-425 (1999).

[14] H. Goldstein, C. Poole and J. Safko *Classical Mechanics*, Third Edition, (Addison-Wesley, San Francisco 2002). We use the canonical transformations scheme followed here. In the first and second editions of this widely used text the period of the relativistic harmonic oscillator, obtained by direct integration, has a sign error, incorrectly showing an increase in period from

its non-relativistic value. This error persists in the first six printings of the third edition, and has been corrected in subsequent printings.

[15] M.K.Balasubramanya,"Two forms of the action variable for the relativistic harmonic oscillator,", arXiv: math-ph/0702036.

Tables

TABLE I: Action variable for a classical relativistic harmonic oscillator from 4 schemes.

Nature of oscillator	**Form of J_C**	**Expression for $J_C(\tilde{E})$**
Fully relativistic case	$\frac{1}{2\pi}\oint p dx$	$\frac{\tilde{E}}{\omega_0}\sqrt{(1+\frac{\epsilon}{2})}\left[1-\frac{1}{8}\left(\frac{\epsilon}{2+\epsilon}\right)-\frac{1}{64}\left(\frac{\epsilon}{2+\epsilon}\right)^2\ldots\right]$
Fully relativistic case	$-\frac{1}{2\pi}\oint x dp$	$\frac{\tilde{E}}{\omega_0}\sqrt{1+\frac{\epsilon}{2}}\ \left[1-\frac{1}{16}\epsilon+\frac{7}{256}\epsilon^2+\frac{1}{128}\epsilon^3\ldots\right]$
Weak relativistic case	$\frac{1}{2\pi}\oint p dx$	$\frac{\tilde{E}}{\omega_0}\ (1+\frac{3}{16}\epsilon)$, to order ϵ
Weak relativistic case	$-\frac{1}{2\pi}\oint x dp$	$\frac{\tilde{E}}{\omega_0}\ (1+\frac{3}{16}\epsilon)$, to order ϵ

TABLE II: Correction to zeroth order energy eigenvalues, $\tilde{E}_{n,0}=(n+\frac{1}{2})\,\hbar\omega_0$, of weakly relativistic harmonic oscillator.

Calculation scheme	**Calculated term**	**Correction to $\tilde{E}_{n,0}$**
Evaluate quantum action variable in $\frac{1}{2\pi}\oint p dx$ form	$J_{WR}=\frac{\tilde{E}}{\omega_0}\ [1+\epsilon\{\frac{3}{16}+\frac{7}{16}\left(\frac{\hbar\omega_0}{\tilde{E}}\right)-\frac{17}{64}\left(\frac{\hbar\omega_0}{\tilde{E}}\right)^2\}]-\frac{\hbar}{2}$	$-\frac{3}{16}\hbar\omega_0\left[(n+\frac{5}{3})^2-(\frac{5}{3})^2\right]\frac{\hbar\omega_0}{mc^2}$
Evaluate quantum action variable in $-\frac{1}{2\pi}\oint x dp$ form	$J_{WR}=\frac{\tilde{E}}{\omega_0}-\frac{\hbar}{2}$ $+\frac{3\hbar}{64}\left\{1+\frac{4\tilde{E}^2}{(\hbar\omega_0)^2}\right\}\frac{\hbar\omega_0}{mc^2}$	$-\frac{3}{16}\hbar\omega_0\left[\left\{n+\frac{1}{2}-\frac{3}{4}(\frac{\hbar\omega_0}{mc^2})\right\}^2+4\right]\frac{\hbar\omega_0}{mc^2}$
JWKB approximation	$J_{CWR}=\frac{\tilde{E}}{\omega_0}\left[1+\frac{3}{16}\epsilon\right]$	$-\frac{3}{16}\hbar\omega_0(n+\frac{1}{2})^2\ \frac{\hbar\omega_0}{mc^2}$
Rayleigh-Schrdinger perturbation theory	$\Delta E_n=<\phi_n\vert-\frac{\hat{p}^4}{8m^3c^2}\vert\phi_n>$	$-\frac{3}{16}\hbar\omega_0[(n+\frac{1}{2})^2+\frac{1}{4}]\ \frac{\hbar\omega_0}{mc^2}$